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ORIGINAL PAPER

Impulsive state feedback control of the microorganism culture in a turbidostat

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Abstract In this paper, a mathematical model with the impulsive state feedback control is proposed for turbidostat system. The sufficient conditions of existence of order-1 and order-2 periodic solutions are obtained by the existence criteria of periodic solution of a general planar impulsive autonomous system. It is shown that the system either tends to a stable state or has a periodic solution, which depends on the feedback state and the initial concentration of microorganism and substrate. Finally, some discussions and numerical simulations are given.

Keywords Turbidostat · Impulsive state feedback control · Period-1 solution · Period-2 solution

1 Introduction

A chemostat is a piece of laboratory apparatus frequently used for culturing microorganisms. It can be used for representing all kinds of microorganism systems such as lake, waste-water treatment and reaches for commercial production of the advantage of being easily implementable in a laboratory. Hence the model has been

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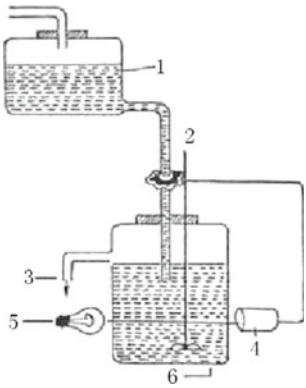

Fig. 1 1. Reservior of sterile Medium 2. Valve controlling flow of medium 3. Outlet for spent medium 4. Photo cell 5. Light source 6. Turbidostat

subject to extensive tests and experiments. In order to investigate the dynamics of the microorganism growth in a chemostat, many papers have investigated the mathematical models on the culture of the microorganisms. For example, mathematical [1–6] and experimental [7–9] models exhibit the competitive exclusion principle only one species survives. Several modifications of the chemostat have been made to ensure the coexistence of species on a single nutrient [7,9–11].

There are many aspects to be considered in order to achieve higher biomass concentration and productivity. With the growth of the microorganism and its concentration increasing in the chemostat, the effect of inhibition between the production and other negative effect will occur when the concentration of the microorganism reaches a critical value. Therefore, how to control the microorganism concentration is important to decrease the inhibition of the microorganism concentration. For the purpose of continuously culturing the microorganism and decreasing the inhibition effect or other negative effects, it is necessary to keep the microorganism concentration lower than the critical level. So the chemostat with the feedback control of the dilution rate, which is often referred to as a turbidostat by bio-engineers and biologists [12], is established. In this sense, the turbidostat is actually a continuous culture system of the microorganism which includes an optical sensing device (optoelectronic device) by which the concentration of the microorganism in the growth vessel and the dilution rate can be controlled(see Fig. 1). In the turbidostat, an optical sensor measures the turbidity of the fluid, which is used to control the dilution rate [12]. If the culture density becomes too high the dilution rate increases, on the contrary if it becomes too low the dilution rate decreases.

There are some papers investigating the mathematical models on the turbidostat by means of feedback control of the dilution rate [10–12]. The coexistence of two species in the turbidostat was shown numerically by Flegr [10], and later analytically by De Leenheer and Smith [11].

Recently, theories of impulsive differential equations have been introduced into population dynamics. Especially, impulsive state feedback control strategy is used widely in real life problems [13–15]. For example, Yang [16] presents models of impulsive electronic devices, which are ideal models of nanoelectronic devices, and studies some examples of nanoelectronic circuits consisting of driven single-electron tunneling junctions. To make the rocket transfer to a higher energy orbit, increments in velocity are given impulsively when the rocket reaches the position of peri-apse and apo-apse [17].

In real life, we use the impulsive state feedback control to investigate the micro-organism culture, instead of impulsive differential equations with the fixed moment because the control measures are taken only the microorganism concentration reaches a threshold value. Based on the above ideas [10–17], we will introduce the impulsive state feedback control into the turbidostat.

An outline of this paper is as follows: an autonomous system with the impulsive state feedback strategy is introduced into the turbidostat in Sect. 2. In addition, some definitions and existence criteria of the periodic solution for a general planar impulsive autonomous system are also given in Sect. 2. In Sect. 3, the qualitative analysis is given and the existences of order one and order two periodic solutions are investigated. Finally, we give numerical simulations and a brief discussion.

2 Model description and preliminaries

The basic deterministic models of microbial growth in the continuous culture apparatus take the following form:

$$\begin{cases} \frac{dS'}{dt'} = QS^0 - QS' - \frac{\mu S'x'}{\delta(kx'+S')}, \\ \frac{dx'}{dt'} = \frac{\mu S'x'}{kx'+S'} - Qx', \end{cases} \tag{2.1}$$

where $S'(t')$ and $x'(t')$ denote the concentration of the substrate and microorganism, respectively at time t'. Q is the dilution rate and S^0 is the concentration of the input substrate. μ is called the maximal specific growth rate of the microorganism. k is the positive constant. The yield constant δ reflects that only a fraction of the nutrient of what the species consumes, leads to new biomass.

The microorganism concentration in the culture vessel is lower than a threshold value (a predetermined value by experiment), it is not necessary to take a control measure. While the microorganism concentration reaches the critical value which may be detected by the optoelectronic devices, the control measure should be taken to decrease the microorganism concentration (see Fig. 2).

The feedback approach is perhaps most natural in the lab setting. For instance, optical sensors can be used to measure turbidity, giving a rough estimate of the concentrations of the species. The concentration estimate can be processed by a computer to (online) calculate the dilution rate. The result then determines the speed of the pump-the device that is being actuated-which supplies the reactor with fresh medium.

We introduce the impulsive state feedback control into the system (2.1). Then system (2.1) becomes as follows:

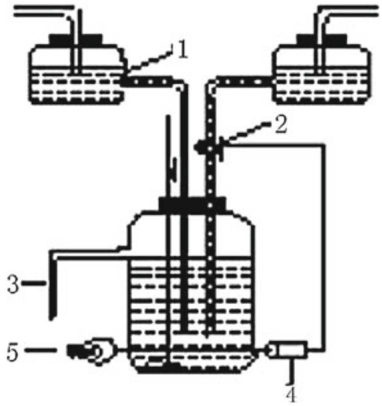

Fig. 2 1. Reservior of sterile medium 2. Valve controlling flow of medium 3. Outlet for spent medium 4. Photo cell 5. Light source

$$\begin{cases} \left.\begin{aligned} \frac{dS'}{dt'} &= QS^0 - QS' - \frac{\mu S'x'}{\delta(kx'+S')}, \\ \frac{dx'}{dt'} &= \frac{\mu S'x'}{kx'+S'} - Qx', \end{aligned}\right\} & x' < x_h, \\ \left.\begin{aligned} \Delta S' &= -Q_1 S', \\ \Delta x' &= -Q_1 x', \end{aligned}\right\} & x' = x_h, \\ S'(0) = S'_0, x'(0) = x'_0, \end{cases} \tag{2.2}$$

where $\Delta S'(t) = S'(t^+) - S'(t)$, $\Delta x'(t) = x'(t^+) - x'(t)$. Q_1 is a dimensionless feedback coefficient due to the feedback control when the microorganism concentration x' reaches the critical value x_h (see Fig. 2). Other parameters are the same as the system (2.1).

The variables in the above system may be rescaled by measuring $S = \frac{S'}{S^0}$, $x = \frac{x'}{\delta S^0}$, $t = Qt'$, then system (2.2) becomes

$$\begin{cases} \left.\begin{aligned} \frac{dS}{dt} &= 1 - S - \frac{mSx}{ax+S}, \\ \frac{dx}{dt} &= \frac{mSx}{ax+S} - x, \end{aligned}\right\} & x < h, \\ \left.\begin{aligned} \Delta S &= -pS \\ \Delta x &= -px, \end{aligned}\right\} & x = h, \\ S(0) = S_0, x(0) = x_0, \end{cases} \tag{2.3}$$

where

$$m = \frac{\mu}{Q}, \quad a = k\delta, \quad h = \frac{x_h}{\delta S^0}, p = Q_1.$$

In the following, we will discuss the existence of periodic solution of (2.3) by the existence criteria of the general impulsive autonomous system. For late convenience, we give some definitions and lemmas.

Definition 2.1 [18] An triple (X, π, R_+) is said to a semi-dynamical system if X is a metric space, R_+ is the set of all non-negative reals and $\pi : X \times R_+ \to X$ is a continuous function such that

(i) $\pi(x, 0) = x$ for all $x \in X$;

(ii) $\pi(\pi(\pi, t), s) = \pi(x, t + s)$ for all $x \in X$ and $t, s \in R_+$.

We denote a semi-dynamical system (X, π, R_+) by (X, π). For any $x \in X$, the function $\pi_x : R_+ \to X$ defined as $\pi_x(t) = \pi(x, t)$ is continuous and we call π_x the trajectory of x. The set $C^+(x) = \{\pi(x, t)|t \in R_+\}$ is called the positive orbit of x. For any subset M of X, we let $M^+(x) = C^+(x) \cap M - x$ and $M^-(x) = G(x) \cap M - x$, where $G(x) = \cup\{G(x, t)|t \in R_+\}$ and $G(x) = \{y|\pi(y, t) = x\}$ is a attainable set of x at $t \in R_+$. Finally, we set $M(x) = M^+(x) \cup M^-(x)$.

Definition 2.2 [18] An impulsive semi-dynamical system (X, π, M, I) consists of a semi-dynamical system (X, π) together with a nonempty closed subset M of X and a continuous function $I : M \to X$ such that the following properties:

(i) No point $x \in X$ is a limit point of $M(x)$.

(ii) $\{t|G(x, t) \cap M \neq \emptyset\}$ is a closed subset of R_+.

We write $N = I(M) = \{y \in X|y = I(x), x \in M$ and for any $x \in X, I(x) = x^+\}$. We call M the set of impulses and I the impulsive function.

Defining a function $\Phi : X \to R_+ \cup \{\infty\}$ as follows:

$$\Phi(x) = \begin{cases} \infty & \text{if} \quad M^+(x) = \emptyset, \\ s & \text{if} \quad \pi(x, t) \notin M \quad \text{for} \quad 0 < t < s \quad \text{and} \quad \pi(x, s) \in M, \end{cases}$$

where s is called the time without impulse of x. i.e. s is the first time when $\pi(x, 0)$ hits M.

Definition 2.3 [18] Let (X, π, M, I) be an impulsive semi-dynamical system and $x \in X$ and $x \notin M$. The trajectory of x is a function $\widetilde{\pi}_x$ defined on subset $[0, s)$ of R_+ (s mat be ∞) to X inductively as follows:

$$\widetilde{\pi}_x = \widetilde{\pi}(\widetilde{x}_{n-1}, t), \ \tau_{n-1} < t < \tau_n,$$

where x_n is the sequence of impulse point of x, which satisfies $\pi(x^+_{n-1}, \Phi(x^+_{n-1})) = x_n$. τ_n is the sequence of impulsive time relative to $\{x_n\}$, $\tau_n = \sum_{k=0}^{n-1} \Phi(x^+_k)$.

Definition 2.4 [18] A trajectory $\widetilde{\pi}_x$ is said to be periodic of period τ and order k if there exist positive integers $m \geq 1$ and $k \geq 1$ such that k is the smallest integer for $x^+_m = x^+_{m+k}$ and $\tau = \sum_{i=m}^{m+k-1} \Phi(x^+_i)$.

Consider the following general autonomous impulsive differential equations:

$$\begin{cases} \left.\begin{aligned} \frac{dx}{dt} = P(x, y), \\ \frac{dy}{dt} = Q(x, y), \end{aligned}\right\} (x, y) \notin M, \\ \left.\begin{aligned} \Delta x = I_1(x, y), \\ \Delta y = I_2(x, y), \end{aligned}\right\} (x, y) \in M, \end{cases} \tag{2.4}$$

where $(x, y) \in R^2$. P, Q, I_1 and I_2 are all functions mapping R^2 into R. $M \subset R^2$ is the set of impulse, and we assume:

(H2.1) $P(x, y)$ and $Q(x, y)$ are all continuous with respect to $x, y \in R^2$.

(H2.2) $M \subset R^2$ is a linear function, $I_1(x, y)$ and $I_2(x, y)$ are linear functions of x and y.

For each point $S(x, y) \in M$, we define $I : R^2 \to R^2$:

$$I(S) = (x^+, y^+) \in R^2, x^+ = x + I_1(x, y), y^+ = y + I_2(x, y).$$

Obviously, $N = I(M)$ is also a linear function of R^2 or a subset of a line and we assume $M \cap N = \emptyset$. From Definition 2.2, we know system (2.4) is an impulsive semi-dynamical system. The following lemma gives the conditions under which system (2.4) has a periodic solution of order one by Definition 2.4.

Lemma 2.5 [19] *If system* (2.4) *satisfies assumptions (H2.1) and (H2.2), and there exists a boundedly closed and simply connected region D which has the following properties:*

(i) *There is no singularity in it and the boundary ∂D is composed of three parts: L_1, L_2 and L_3.*

(ii) *$L_1 = D \cap M$ cannot be tangent with trajectories of* (2.4) *except at end-points.*

(iii) *$L_2 \subset I(M)$ is a line segment which satisfies $I(L_1) \subset L_2$.*

(iv) *Trajectories with initial point in $L_2 \cup L_3$ will enter into the interior of D, then there must exist a periodic solution of order one in region D.*

3 Existence of periodic solution of system (2.3)

Before discussing the periodic solution of system (2.3), we should consider the qualitative characteristic of (2.3) without the impulsive effect. Then the corresponding system without the impulsive affection is as follows:

$$\begin{cases} \dot{S}(t) = 1 - S - \frac{mSx}{ax+S}, \\ \dot{x}(t) = \frac{mSx}{ax+S} - x. \end{cases} \tag{3.1}$$

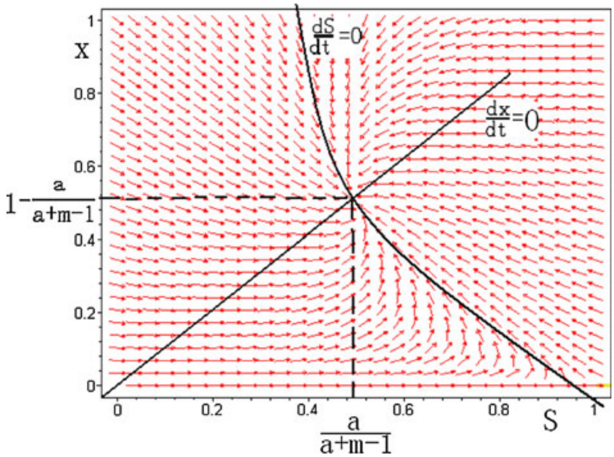

Fig. 3 Vector graph of system (3.1) when $m > 1$

By simple calculation, system (3.1) has a microorganism-free equilibrium $(1, 0)$ and a positive equilibrium $\left(\frac{a}{a+m-1}, 1 - \frac{a}{a+m-1}\right)$, $a + m > 1$.

Obviously, the following results about system (3.1) can be obtained.

(1) Every solution of system (3.1) tends to $(1, 0)$ if $m < 1$ holds.

(2) If $m > 1$ holds, then $(1, 0)$ is a saddle point, the positive equilibrium $\left(\frac{a}{m+a-1}, 1 - \frac{a}{m+a-1}\right)$ is globally asymptotically stable node in the positive quadrant($S(t) > 0, x(t) > 0$) and

$$\lim_{t\to\infty} S(t) = \frac{a}{m+a-1}, \quad \lim_{t\to\infty} x(t) = 1 - \frac{a}{m+a-1}.$$

The vector graph of system (3.1) can be seen in Fig. 3. For the initial points which satisfy $x(0) < 1 - \frac{a}{m+a-1}$ and $\frac{dS}{dt}|_{(S_0,x_0)} \geq 0$, if $h > 1 - \frac{a}{m+a-1}$, then all the solutions of system (2.3) tend to the equilibrium $\left(\frac{a}{m+a-1}, 1 - \frac{a}{m+a-1}\right)$ and no impulse will occur, which indicates the microorganism concentration will not affect the microorganism culture. Therefore, we need not control the concentration of the microorganism. In fact, anaerobic microorganisms such as anaerobic bacteria lactococcus, an l-lactic acid bacteria and ethanol producing bacteria are unlike common aerobic industrial microorganisms such as yeast for ethanol fermentation and bacteria for glutamic acid fermentation. Serious end product (L-Lactic acid) inhibition was observed in the microorganism as described as in the previous report [20]. So we mainly focus our attention on the case $h < 1 - \frac{a}{m+a-1}$, $x(0) < 1 - \frac{a}{m+a-1}$ and $S(0) \leq 1$.

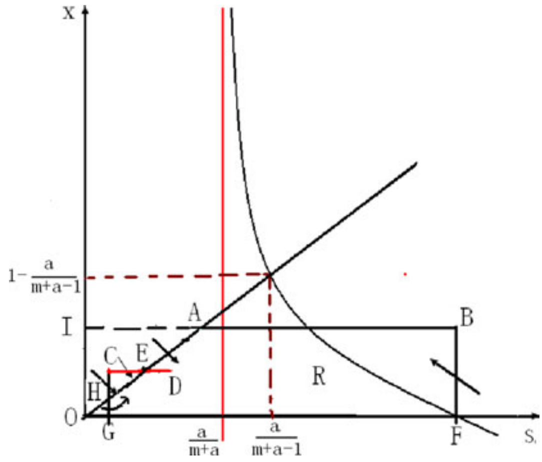

Fig. 4 Existence of periodic solution of order one when $S_C < \frac{ah}{m-1}$ and $S_{DM} > S_D > \frac{ah}{m-1}$

3.1 Order one periodic solution

In Fig. 4, the line $x = h$ intersects the isoclinal line $\mathrm{d}x/\mathrm{d}t = 0$ at the point (S_A, h), where $S_A = \frac{ah}{m-1}$. The impulsive set $M \subseteq \overline{AB}, \overline{AB} = \{(S, x)|x = h, S_A \le S \le 1\}$. The impulsive functions I_1 and I_2 map the impulsive set M as $N = I(M) \subseteq \overline{CD}, \overline{CD} = \{x = (1-p)h, (1-p)S_A \le S \le 1-p\}$, where $C = (S_C, (1-p)h)$, $S_C = (1-p)\frac{ah}{m-1}$, $S_D = 1-p$. From the third equation of (2.3), we know that $S^+ = (1-p)S$ for $x = h$ and furthermore $S_C = (1-p)S_A \le S_A$. According to the value of S_C and S_D, we mainly discuss the following cases:

Case I: $S_C \le \frac{ah}{m-1}$ and $\frac{ah}{m-1} \le S_D \le S_{DM}$, where

$$1 - S_{DM} - \frac{mh(1-p)S_{DM}}{ah(1-p) + S_{DM}} = 0,$$

$$S_{DM} = \frac{-(ah(1-b) + mh(1-p) - 1) + \sqrt{(ah(1-p) + mh(1-b) - 1)^2 + 4ah(1-p)}}{2}.$$

Case II: $S_C < \frac{ah}{m-1}$ and $S_D < \frac{ah}{m-1}$ (Fig. 5).
Case III: $S_C < \frac{ah}{m-1}$ and $S_D > S_{DM}$ (Fig. 6).

We first discuss Case I, $S_C \le \frac{ah}{m-1}$ and $\frac{ah}{m-1} \le S_D \le S_{DM}$, the illustration can be seen in Fig. 4. From the qualitative characteristic of (3.1), it is easily known that all the trajectories of (2.3) starting from the region $\{x(0) > h, \frac{dS}{dt}|_{(S(0),x(0))} \le 0\}$ do not interact the line $x = h$ and the trajectories starting from the region $\{x(0) < h, \frac{dS}{dt}|_{(S(0),x(0))} \le 0\}$ must interact with the segment $\overline{AB}$.

Theorem 3.1 *Suppose that* $m > 1$, $h < 1 - \frac{a}{m+a-1}$, $S_C \le \frac{ah}{m-1} \le S_D \le S_{DM}$ *and* $x(0) < h$, *then system* (2.3) *has an order one periodic solution.*

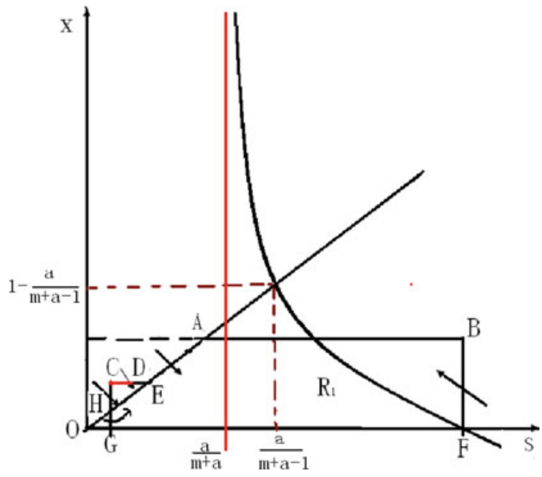

Fig. 5 Existence of periodic solution of order one when $S_C < \frac{ah}{m-1}$ and $S_D < \frac{ah}{m-1}$

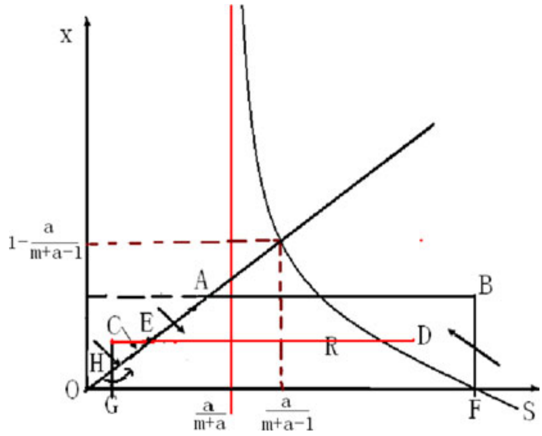

Fig. 6 Existence of periodic solution of order one when $S_C < \frac{ah}{m-1}$ and $S_D > S_{DM}$

Proof In order to apply to Lemma 2.5, we construct a closed region R (see Fig. 4). We know that $\frac{dS}{dt} < 0$, $\frac{dx}{dt} > 0$ for $S = 1$ and $\frac{dS}{dt} = 0$ for $x = 0$. Furthermore, the perpendicular line $S = S_C = (1-p)S_A$ interacts the axis $x = 0$ at the point G and interacts the line $x = \frac{aS}{m-1}$ at the point H. According to the qualitative property of system (2.3), we obtain $\frac{dS}{dt}|_{\overline{CH}} > 0$, $\frac{dx}{dt}|_{\overline{CH}} < 0$ and $\frac{dS}{dt}|_{\overline{HG}} > 0$, $\frac{dx}{dt}|_{\overline{HG}} > 0$. The segment $\overline{CD}$ interacts the straight line $x = \frac{m-1}{a}S$ at the point E, we have $\frac{dS}{dt}|_{\overline{AE}} > 0$, $\frac{dx}{dt}|_{\overline{AE}} < 0$, $\frac{dS}{dt}|_{\overline{CE}} > 0$, $\frac{dx}{dt}|_{\overline{CE}} < 0$. According to Lemma 2.5, we can obtain that system (2.3) has an order one periodic solution.

Next we consider Case II.

Theorem 3.2 *Suppose that* $m > 1$, $h < 1 - \frac{a}{m+a-1}$, $S_C \leq \frac{ah}{m-1}$, $S_D \leq \frac{ah}{m-1}$ *and* $x(0) < h$ *then system* (2.3) *has an order one periodic solution.*

Proof For Case II, we easily obtain the closed region R_1 (see Fig. 5). We extend the segment $\overline{CD}$ to interact the segment $\overline{AE}$ at the point E. Since $\frac{dS}{dt}|_{\overline{CE}} > 0$, $\frac{dx}{dt}|_{\overline{CE}} < 0$.

Similar to Case I, the closed region R_1 consists of $\overline{CG}$, $\overline{GF}$, $\overline{FB}$, $\overline{BA}$, $\overline{AE}$, $\overline{ED}$, $\overline{DC}$. It follows from Lemma 2.5 that system (2.3) has an order one periodic solution.

Theorem 3.3 *Suppose that* $m > 1$, $h < 1 - \frac{a}{m+a-1}$, $S_C \leq \frac{ah}{m-1}$, $S_D > S_{DM}$ *and* $x(0) < h$ *then system* (2.3) *has an order one periodic solution.*

The proof is similar to Theorem 3.1 and 3.2, we omit it.

Next we give one lemma firstly to discuss the stability of this positive periodic solution of system (2.3).

Lemma 3.4 *The T-periodic solution* $S(t) = \xi(t), x(t) = \eta(t)$ *of the system*

$$\begin{cases} \left.\begin{aligned} \frac{dS}{dt} = P(S, x), \\ \frac{dx}{dt} = Q(S, x), \end{aligned}\right\} \phi(S, x) \neq 0, \\ \left.\begin{aligned} \Delta S = \alpha(S, x), \\ \Delta x = \beta(S, x), \end{aligned}\right\} \phi(S, x) = 0, \end{cases} \tag{3.2}$$

is orbitally asymptotically stable if the Floquet multiplier μ_2 satisfies the condition $|\mu_2| < 1$, where

$$\mu_2 = \Pi_{k=1}^{q} \Delta_k \exp\left[\int_0^T \left(\frac{\partial P}{\partial S}(\xi(t), \eta(t)) + \frac{\partial Q}{\partial x}(\xi(t), \eta(t))\right) dt\right], \tag{3.3}$$

with

$$\Delta_k = \frac{P_+\left(\frac{\partial \beta}{\partial y}\frac{\partial \phi}{\partial x} - \frac{\partial \beta}{\partial x}\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial x}\right) + Q_+\left(\frac{\partial \alpha}{\partial x}\frac{\partial \phi}{\partial y} - \frac{\partial \alpha}{\partial y}\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\right)}{P\frac{\partial \phi}{\partial x} + Q\frac{\partial \phi}{\partial y}}$$

and P, Q, $\frac{\partial \alpha}{\partial x}$, $\frac{\partial \alpha}{\partial y}$, $\frac{\partial \beta}{\partial x}$, $\frac{\partial \beta}{\partial y}$, $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial y}$ are calculated at the point $(\xi(\tau_k), \eta(\tau_k))$, $P_+ = P(\xi(\tau_k^+), \eta(\tau_k^+))$, $Q_+ = Q(\xi(\tau_k^+), \eta(\tau_k^+))$. $\phi(x, y)$ is a sufficiently smooth function with grad$\phi(x, y) \neq 0$ and $\tau_k (k \in N)$ is the time of kth jump.

The proof of this lemma is referred to [21].

In what follows, we suppose this periodic solution of system (2.3) with period T passes through the points $E^+(\xi_0(1-p), (1-p)h)$ and (ξ_0, h). As the expression and the period of this solution are unknown, we discuss the stability of this positive periodic solution by using Lemma 3.4. In our case,

$$\begin{aligned} P(S, x) &= 1 - S - \frac{mSx}{ax + S}, \quad Q(S, x) = \frac{mSx}{ax + S} - x, \alpha(S, x) \\ &= -pS, \beta(S, x) = -px, \end{aligned}$$

$$\phi(S, x) = x - h, (\xi(T), \eta(T)) = (\xi_0, h), \quad (\xi(T^+), \eta(T^+)) = ((1-p)\xi_0, (1-p)h).$$

$$\frac{\partial P}{\partial S} = -1 - \frac{max^2}{(ax + S)^2}, \quad \frac{\partial Q}{\partial x} = \frac{mS^2}{(ax + S)^2} - 1,$$

$$\frac{\partial \alpha}{\partial S} = -p, \frac{\partial \alpha}{\partial x} = 0, \frac{\partial \beta}{\partial S} = 0, \frac{\partial \beta}{\partial x} = -p, \frac{\partial \phi}{\partial S} = 0, \frac{\partial \phi}{\partial x} = 1.$$

$$\begin{aligned}
\Delta_1 &= \frac{P_+\left(\frac{\partial \beta}{\partial x}\frac{\partial \phi}{\partial S} - \frac{\partial \beta}{\partial S}\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial S}\right) + Q_+\left(\frac{\partial \alpha}{\partial S}\frac{\partial \phi}{\partial x} - \frac{\partial \alpha}{\partial x}\frac{\partial \phi}{\partial S} + \frac{\partial \phi}{\partial x}\right)}{P\frac{\partial \phi}{\partial S} + Q\frac{\partial \phi}{\partial x}} \\
&= \frac{Q_+(\xi(T^+), \eta(T^+)(1-p)}{Q(\xi(T), \eta(T))} \\
&= \frac{(1-p)^2 h\left(\frac{m(1-p)\xi_0}{a(1-p)h+(1-p)\xi_0} - 1\right)}{h\left(\frac{m\xi_0}{ah+\xi_0} - 1\right)} = (1-p)^2.
\end{aligned}$$

Set $G(t) = \frac{\partial P}{\partial S}(\xi(t), \eta(t)) + \frac{\partial Q}{\partial x}(\xi(t), \eta(t))$, then

$$\begin{aligned}
\mu_2 &= \Delta_1 \exp\left(\int_0^T \left(\frac{\partial P}{\partial S}(\xi(t), \eta(t)) + \frac{\partial Q}{\partial x}(\xi(t), \eta(t))\right) dt\right) \\
&= (1-p)^2 \exp\left(\int_0^T G(t)dt\right).
\end{aligned}$$

If $|\mu_2| < 1$, that is

$$|(1-p)^2 \exp\left(\int_0^T G(t)dt\right)| < 1,$$

the periodic solution is stable. Thus the result about the existence and stability of this positive period-1 solution is given in the following:

Theorem 3.5 *Suppose that* $m > 1$, $h < 1 - \frac{a}{m+a-1}$ *and* $x(0) < h$ *then system* (2.3) *has an order one periodic solution. Furthermore, this period-*1 *solution is stable if* $|(1-p)^2 \exp(\int_0^T G(t)dt)| < 1$, *where* $G(t) = \frac{m(\xi(t)^2 - a\eta(t)^2)}{(a\eta(t)+\xi(t))^2} - 2$.

3.2 The existence of order two periodic solution

To discuss the dynamics of system (2.3), we choose two sections $X_0 = \{(S, x)|S \geq 0, x = (1-p)h\}$ and $X_1 = \{(S, x)|S \geq 0, x = h\}$ to establish a Poincare map. Suppose the point $B_k(S_k, h)$ is on the Poincare section X_1. Then $B_k^+((1-p)S_k, (1-p)h)$ is on X_0 due to impulsive effects, and the trajectory with the initial point B_k^+ intersects the Poincare section X_1 at the point $B_{k+1} = (S_{k+1}, h)$, where S_{k+1} is decided by S_k and the parameter p. We get the following Poincare map

$$S_{k+1} = F(p, S_k). \tag{3.4}$$

From the dependence of the solutions on the initial conditions, the function is continuous on p and S_k. For each fixed point of the Poincare map, there is an associated periodic solution of system (2.3), and vice versa.

From Theorem 3.1 to Theorem 3.4, we know that system (2.3) has an order one periodic solution. In this section, we will discuss order two periodic solution.

Suppose that $(\widetilde{S}(t), \widetilde{x}(t))$ is a periodic solution of system (2.3), then $(\widetilde{S}_0', (1-p)h) \in N \subseteq \overline{CD}$ and $(\widetilde{S}_0, h) \in M \subseteq \overline{AB}$. It is easily obtained that $\widetilde{S}_0' < \widetilde{S}_0$ due to the impulsive effect.

Let $(S_0', x_0') \in N \subseteq \overline{CD}$ and $(S_0, x_0) \in M \subseteq \overline{AB}$. It is also easily obtained that $S_0' < S_0$ since $S(t^+) = (1-p)S(t)$ by third equation of system (2.3). Here, we denote the arbitrary solution of system (2.3) by $(S(t), x(t))$. The second interaction point of the trajectory and set $M(x = h)$ is denoted as the point (S_1, h). After a series of impulse, the corresponding interaction points of trajectory and set M are $(S_i, h), i = 3, 4, \ldots$. From the Poincare map (3.4), we have $S_1 = F(p, S_0)$, $S_2 = F(p, S_1)$ and $S_{n+1} = F(p, S_n)(n = 3, 4, \ldots)$.

By the qualitative analysis of system (2.3), we know that $\frac{dx}{dt} < 0$ for the region $(S, x) \in \overline{AOIA}$ and $\frac{dx}{dt} > 0$ for the region $(S, x) \in \overline{ABFOA}$ (see Fig. 4). So we consider the following cases:

Case 1 If the periodic solution is in the region $\overline{ABFOA}$. The trajectory starting from the point $(S_0', (1-p)h)$ will interact the segment $\overline{AB}$ at the point (S_0, h) and then jumps to $(S_0^+, (1-p)h)$. The trajectory starting from the point $((S_0^+, (1-p)h))$ will interact the segment $\overline{AB}$ at the point (S_1, h). The trajectory starting from the point $(S_1^+, (1-p)h)$ will again interact the segment at the point (S_2, h) and so on. Without loss of generality, we suppose that $S_0 < \widetilde{S}_0$. By the qualitative property of the system (2.3) in the region $\overline{ABFOA}$, one and only one of the following sequences holds:

(A):

$$S_0 \leq S_1 \leq S_2 \leq S_3 \leq \cdots \leq \widetilde{S}_0,$$

(B):

$$\widetilde{S}_0 \geq S_0 \geq S_1 \geq S_2 \geq S_3 \geq \cdots .$$

It is known that the sequences the trajectories tend to be periodic since the sequences are monotone and ultimately bounded. It follows by Definition 2.4 that order two periodic solution does not exist in this case.

Case 2 If the periodic solution is in the region $\overline{AOIA}$ (see Fig. 4), then we have $\frac{dx}{dt} < 0$. For any two points $B_m(S_m, h)$ and $B_j(S_j, h)$ in the region $\overline{AOIA}$, where $S_m < S_j$. We have $S_m^+ = (1-p)S_m$ and $S_j^+ = (1-p)S_j$ due to impulsive effect. Then it follows from the vector field of system (2.3) that $0 < S_{j+1} < S_{m+1} < 1$, that is

$$0 < S_{j+1} < S_{m+1} < 1 \quad \text{for} \quad 0 < S_m < S_j < 1. \tag{3.5}$$

From the poincare map (3.4) we have $S_1 = F(p, S_0)$, $S_2 = F(p, S_1)$, and $S_{n+1} = F(p, S_n)(n = 3, 4 \ldots)$.

1. If $S_0 = S_1$, then system (2.3) has a positive period-1 solution;
2. If $S_0 \neq S_1$, without loss of generality, suppose that $S_1 < S_0$. It follows from (3.5) that $S_2 > S_1$. Furthermore, if $S_2 = S_0$, then system (2.3) has a positive period-2 solution.
3. If $S_0 \neq S_1 \neq S_2 \neq S_3 \neq \cdots S_{k-1} (k \geq 3)$ and $S_0 = S_k$, then system (2.3) has a positive period-k solution. In fact, this is impossible. If $S_0 < S_1$ then from (3.5), we have $S_1 > S_2$ and then $S_2 < S_0 < S_1$ or $S_0 < S_2 < S_1$. If $S_0 > S_1$, then from (3.5), we have $S_1 < S_2$ and then $S_1 < S_2 < S_0$ or $S_1 < S_0 < S_2$. So the relation of S_0, S_1 and S_2 is one the following:

$$S_2 < S_0 < S_1, \quad S_0 < S_2 < S_1, \quad S_1 < S_2 < S_0, \quad S_1 < S_0 < S_2.$$

(i) $S_2 < S_0 < S_1$
If $S_2 < S_0 < S_1$ holds, then from (3.5), we have $S_3 > S_1 > S_2$. It is also true that $S_3 > S_1 > S_0 > S_2$. We again obtain $S_4 < S_2 < S_1 < S_3$ and then $S_4 < S_2 < S_0 < S_1 < S_3$. By means of induction, we have

$$\begin{aligned} 0 &< \cdots < S_{2k} < \cdots S_4 < S_2 < S_0 < S_1 < S_3 < S_5 \\ &< \cdots < \cdots < S_{2k+1} < \cdots < 1. \end{aligned} \tag{3.6}$$

Similar to (i), we have

(ii) $S_0 < S_2 < S_1$

$$\begin{aligned} 0 &< S_0 < S_2 < S_4 < \cdots < S_{2k} < \cdots < S_{2k+1} < \cdots < S_5 \\ &< S_3 < S_1 < 1. \end{aligned} \tag{3.7}$$

(iii) $S_1 < S_2 < S_0$

$$\begin{aligned} 0 &< S_1 < S_3 < S_5 < \cdots < S_{2k+1} < \cdots < S_{2k} < \cdots < S_4 < S_2 \\ &< S_0 < 1. \end{aligned} \tag{3.8}$$

(iv) $S_1 < S_0 < S_2$

$$\begin{aligned} 0 &< \cdots < S_{2k+1} < \cdots < S_5 < S_3 < S_1 < S_0 < S_2 < S_4 < \cdots < S_{2k} \\ &< \cdots < 1. \end{aligned} \tag{3.9}$$

If there exists a period-k solution ($k \geq 3$) in system (2.3), then we have

$$S_0 \neq S_1 \neq S_2 \neq S_3 \neq \cdots S_{k-1}, \quad S_k = S_0,$$

which is a contradiction to (3.6)–(3.9). So there exists no period-k solution ($k \geq 3$) in system (2.3). In fact, there exists stable period-1 or period-2 solution in this case.

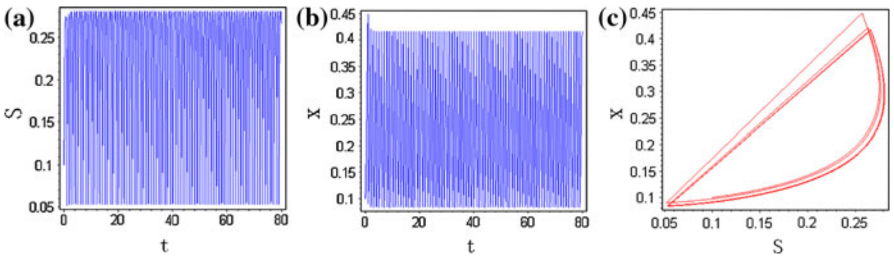

Fig. 7 The time series and portrait phase of the order one periodic solution with the parameters $a = 1, m = 5, p = 0.8, h = 1.6, S(0) = 0.2, x(0) = 0.4$

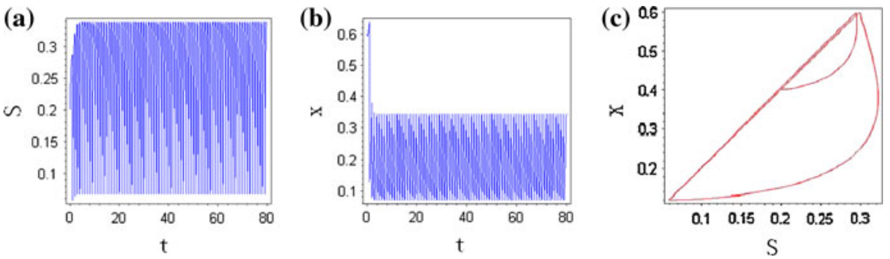

Fig. 8 The time series and portrait phase of the order two periodic solution with the parameters $a = 2, m = 6, p = 0.8, h = 0.8, S(0) = 0.2, x(0) = 0.6$

It follows from (3.6) that $\lim_{n\to\infty} S_{2k} = S_0^*$ and $\lim_{n\to\infty} S_{2k+1} = S_1^*$, where $0 < S_0^* < S_1^* < 1$. Thus $S_1^* = F(p, S_0^*)$ and $S_0^* = F(p, S_1^*)$. So system (2.3) has a stable period-2 solution in the case (i). Similarly, we obtain that system (2.3) has a stable period-1 solution in the case (ii) and (iii) and has a period-2 solution in the case (iv)

4 Discussion

End product inhibition such as L-Latic acid bacteria and ethanol producing bacteria is widely acknowledged in all enzymatic reactions. Therefore, there is a strong economic incentive to develop efficient control strategies that would enable rapid startup and stabilization of steady states in the bioreactors subject to the inhibition of the concentration.

In this paper, we have investigated the existence of order one and order two periodic solutions in a turbidostat model with the impulsive stat feedback control. By using the analogue of Poincare criterion and the Poincare map, we have proved the stability of order one periodic solution, which is simulated in Fig. 7, where $a = 1, m = 5, p = 0.8, h = 1.6, S(0) = 0.2, x(0) = 0.4$. At the same time, we also obtain the existence of the order two periodic solution, which is demonstrated in Fig. 8, where $a = 2, m = 6, p = 0.8, h = 0.8, S(0) = 0.2, x(0) = 0.6$. Now we can give an theoretic analysis on the inhibition of the microorganism concentration. We suppose the predetermined inhibition value of microorganism concentration is 0.8. From Fig. 9b, we know that the microorganism product will ultimately reach the critical value 0.8 and will not again increase even if the substrate concentration is added into the culture

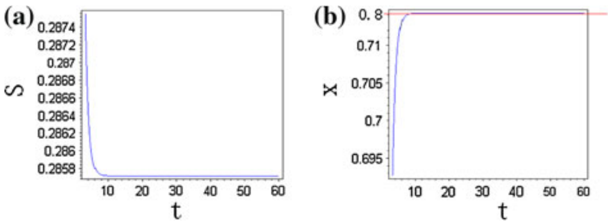

Fig. 9 The phase portrait of the periodic solution. **a** Time-series of the substrate S of system (3.1) with the parameters $a = 2, m = 6, p = 0, S(0) = 0.2, x(0) = 0.6$. **b** Time-series of the microorganism x of system (3.1) with the parameters $a = 2, m = 6, p = 0, S(0) = 0.2, x(0) = 0.6$. **c** The phase portrait of the periodic solution of the impulsive state feedback control of the corresponding system (3.1) with the parameters $a = 2, m = 6, p = 0.8, h = 0.8$

vessel, which implies the microorganism concentration has a great inhibition on the product. In Fig. 9c, when the microorganism concentration reaches some predetermined point $h = 0.8$, fresh water is added to the culture and an equal volume is removed so that the microorganism concentration is under the predetermined point, therefore we obtain an order two periodic solution.

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